A NODE CENTERED FINITE VOLUME FORMULATION FOR THE SOLUTION OF OIL – WATER DISPLACEMENTS IN NON-HOMOGENEOUS POROUS MEDIA

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Resumo – A modelagem do escoamento bifásico de óleo e água em meios porosos heterogêneos constitui-se num grande desafio devido às propriedades das rochas tais como, porosidade e permeabilidade poderem variar de maneira abrupta ao longo de um reservatório. Características geológicas complexas são bastante comuns na modelagem de reservatórios. Algumas formulações do método dos volumes finitos (MVF) adequadas para malhas não-estruturadas podem oferecer vantagens quando comparadas com o tradicional método das diferenças finitas (MDF), que é utilizado comumente na simulação de reservatórios, devido à capacidade natural do MVF em com lidar geometrias complexas e adaptação de malhas. No presente trabalho, apresentamos uma formulação do MVF com volumes de controle centrados no nó e com estrutura de dados baseada em arestas que é utilizada para resolver as equações diferenciais parciais resultantes da modelagem do deslocamento imiscível óleo-água em meios porosos. Esta formulação do MVF é similar ao método dos elementos finitos por aresta, quando elementos triangulares ou tetraédricos são utilizados. As equações de fluxo são resolvidas de maneira segregada através do procedimento IMPES (Pressão Implícita – Saturação Explícita). Ao final do artigo, apresentamos alguns exemplos simples com o objetivo de validar a formulação apresentada.

Palavras-Chave: Estrutura de dados por aresta, volumes finitos, escoamento bifásico, meios heterogêneos

Abstract – Modeling the two-phase flow of oil and water in inhomogeneous porous media is a great challenge due to the fact that rock properties, such as porosity and permeability, can vary abruptly throughout the reservoir. Complex geometrical features are quite common in reservoir modeling. Some unstructured finite volume formulations (FV) can offer advantages relative to standard finite difference methods (FD) which are commonly used in reservoir simulation, due to their natural ability to deal with complex geometries and mesh adaptation. In this paper, we present an unstructured edge-based finite volume formulation which is used to solve the partial differential equations resulting from the modeling of the immiscible displacement of oil by water in non-homogeneous porous media. This FV formulation is similar to the edge-based finite element formulation when linear triangular or tetrahedral elements are employed. Flow equations are solved using a fractional flux approach in a segregated manner through an IMPES (IMplicit Pressure – Explicit Saturation) procedure. Finally, we show some simple model examples in order to validate the presented formulation.

Keywords: Edge-based data structure, finite volume formulation, two-phase flows, non-homogeneous media

1. Introduction

The modeling of multiphase flows in porous media is important not only for the management of petroleum reservoirs but also for environmental remediation problems. The classical example of an immiscible two-phase flow through porous media is the displacement of oil by water in oil reservoirs. This model, which is a great simplification of the real behavior of water and hydrocarbons within reservoirs, is still largely used in petroleum industry, being the starting point when one aims to develop accurate numerical procedures for the solution of the partial differential equations associated to the flux flow through porous media.

In the present work, a vertex centered finite volume formulation using median dual control volumes is implemented using an edge-based data structure that is adapted for solving two dimensional two-phase flow problems in an IMPES (IMplicit Pressure Explicit Saturation) procedure. In this technique, a sequential time stepping procedure is used to decouple the equations. This procedure consists basically in solving one equation at a time. First, considering an initial saturation distribution, the pressure equation is solved and then the velocity field is computed. This velocity field can be utilized as an input for the saturation equation and so on. Edge-based formulations are known to be computationally more efficient than their element based counterparts (Luo *et al*., 1995; Lyra *et al*., 2004; Rees *et al*., 2004), both, in terms of memory usage and CPU costs. Besides they allow for the implementation of different types of finite difference discretizations in the context of 2-D and 3-D unstructured meshes. In the finite volume (FV) formulation presented here, diffusive terms are computed in a two loop approach whereby the first derivatives are computed in the first sweep and second derivatives are computed in the second loop, using a more compact stencil (Crumpton *et al*., 1997; Lyra *et al*., 2004; Rees *et al*., 2004; Carvalho *et al*., 2005). In order to account for the convective terms appearing in the saturation equation, we have used an artificial dissipation scheme which is adapted for use on unstructured meshes. Heterogeneities are handled quite naturally using simple harmonic averages.

2. Phisycal and Mathematical Formulations

The flux flow of oil and water in porous media can be mathematically expressed in many different forms (Baoyan *et al*., 2004). In the classical Peaceman's approach (Peaceman, 1977), the immiscible displacement of oil by water is represented by a set of two equations. The pressure equation which describes the pressure field throughout the reservoir is a variable coefficient parabolic (almost elliptic) partial differential equation. The saturation equation is an essentially hyperbolic conservation law in which capillarity introduces small diffusion-like terms. Both equations are related through the total velocity field which is obtained using the Darcy's law. Even though the approach to be used in our work can seem more complex, it is far more useful when one aims for numerical accuracy and efficiency (Peaceman, 1977 and Baoyan *et al*., 2004). Ignoring gravitational effects, Darcy's velocity can be written for phase *i*, as

$$\vec{v}_i = -\underset{\sim}{K}\lambda_i \nabla P \tag{1}$$

In Equation 1, $\underset{\sim}{K}$ is the absolute permeability tensor of the rock, $\lambda_i = k_i/\mu_i$ is the mobility of phase *i* and k_i and μ_i are respectively, the relative permeability and the viscosity of phase *i*. Without loss of generality, if we ignore capillary effects we can write $P = P_o = P_w$ where o and w, stand, respectively, for oil and water. Besides, in this paper we will focus our attention in incompressible flow, i.e, fluid densities and porosities are independent of pressures.

On the other hand, mass conservation equation can be written as

$$-\nabla\left(\rho_i \vec{v}_i\right) + q_i = \frac{\partial\left(\phi\rho_i S_i\right)}{\partial t} \tag{2}$$

In Equation 2, ρ_i is the phase density, q_i denotes sources or sinks and ϕ is the porosity, i.e., fraction of the rock which can be occupied by fluids. The saturation of phase i, S_i, represents the percentage of the available pore volume occupied by this phase. This last definition implies in the following constitutive relation

$$S_o + S_w = 1 \tag{3}$$

Carrying out the differentiation in Equation 3 and combining Equations 1 to 3, we obtain, after some algebraic manipulation, the following system of equations

$$\nabla \cdot \vec{v} = -Q \tag{4}$$

$$\vec{v} = -\underset{\sim}{K}\lambda \nabla P \tag{5}$$

$$\phi \frac{\partial S_w}{\partial t} + \nabla\left(\vec{F}_w(S_w)\right) = Q_w \tag{6}$$

Equation 4 is the elliptic pressure equation, where $Q = Q_w + Q_o$ or $Q_i = \left(q_i / \rho_i\right)$ is the total injection or production specific rate. In Equation 5, $\vec{v} = \vec{v}_o + \vec{v}_w$ is the total velocity field. Equation 6 is the hyperbolic saturation equation, in which the term $\vec{F}_w = f_w \vec{v}$ is the flux function, where, $f_i = \lambda_i / \left(\lambda_o + \lambda_w\right)$ is the fractional flow function that is highly dependent on the water phase saturation. As it can be seen, the pressure and the saturation fields are connected through the total velocity $\vec{v}$ field. We must also use proper boundary conditions for the pressure and the saturation equations. In the present paper, for the so-called internal boundary (i.e., wells), we will only consider specified well pressures, total injection rates and specified well saturations. On external reservoir boundaries null fluxes boundary conditions are utilized.

3. Edge-Based Finite Volume Formulation

Two major problems must be attacked when solving flux flow within porous media. The correct treatment of the transport terms in the saturation equation, and the accurate numerical solution of the pressure-velocity problem, which is complicated by the fact that the tensor permeability field can varies order of magnitude within reservoir formation. In this context, the adequate discretization of the total velocity field is of utmost importance to guarantee mass conservation. In the present work, we have adopted a node/vertex centered median dual finite volume technique, in which most of the coefficients necessary to our computation are associated with the edges of the mesh. Even though, in principle, there is no restriction to the shape of the elements, it must be over emphasized that edge-based approximations are linear preserving (LP), i.e., they reproduce linear fields exactly, only on triangular and tetrahedral meshes. In median-dual FV formulations, the control volumes (CV) are built connecting centroids of the elements to the middle point of the edges that surround a specific node. In node centered schemes, the fluxes are integrated on the dual mesh usually through one or more loops over the edges, and the computational cost is, therefore, proportional to the number of edges of the mesh.

3.1. The Implicit Pressure Equation

As it is well known, edge-based finite volumes do not allow the consistent calculation of elliptic/diffusive terms with one loop over mesh edges, (Lyra *et al.*, 2004; Rees *et al.*, 2004; Carvalho *et al.*, 2005). As stated previously these terms are usually calculated in a two loop strategy, where gradients are computed in the first loop and the Laplacian operator is computed in the second sweep over the edges. The procedure is briefly described next.

Equation 4 is integrated using the Gauss-Green theorem yielding

$$\int_{\Gamma} \vec{v} \cdot \vec{n} \, \partial \Gamma = \int_{\Omega} Q \tag{7}$$

The discrete form of Equation 7 for a generic node I of the mesh can be written as

$$\sum_{L_{I(\Omega)}} \vec{v}^{\,\Omega}_{IJ_L} \cdot \vec{C}_{IJ_L} + \sum_{L_{I(\Gamma)}} \vec{v}^{\,\Gamma}_{IJ_L} \cdot \vec{D}_{IJ_L} = Q_I V_I \tag{8}$$

In Equation 8, V_I is the volume of the CV, Ω represents approximations on the middle of every edge IJ_L of the mesh which is connected to node *I*, Γ refers to boundary edges connected to that node, $\vec{C}_{IJ_L}$ and $\vec{D}_{IJ_L}$ are the areas of the CV faces, and the summation is performed over the edges $\left(L_I\right)$ connected to node *I*.

In order to approximate the mid-edge velocities $\vec{v}^{\,\Omega}_{IJ_L}$ required in Equation 8, multiple discretization techniques could be used. One bad, but obvious choice is to use a simple finite difference approximation using edge values. It can be proved that this compact stencil is a consistent second order approximation for the Laplacian operator, only in orthogonal meshes (Svärd and Nordström, 2003). Another choice is to use a two loop strategy, in which gradients are computed in the first sweep over the edges and the Laplacian operator is computed in the second one, using the previously computed gradients directly. This extended stencil approximation, which was used by some authors, has proved to be an inconsistent FV approximation for the diffusive operator on general meshes and may lead to “checker-boarding” or “odd-even” oscillations on structured quadrilateral meshes (Svärd and Nordström, 2003; Lyra *et al.*, 2004). In order to overcome such weaknesses, the mid-edge gradients and velocities must be computed in an alternative way. As in the two loop approach, we compute gradients in the first loop over the edges and diffusive fluxes are computed in the second loop using a more compact stencil approach (Crumpton *et al.*, 1997; Lyra *et al.*, 2004; Rees at al., 2004; Carvalho *et al.*, 2005). Using this compact approach, Equation 8 is written for all mesh nodes, producing a non-symmetric system of equations which is assembled in two loops over the edges. Further details can be found in Lyra *et al*. (2004), Rees *et al*. (2004) and Carvalho *et al*., (2005).

3.2. The Explicit Saturation Equation

In this work, in order to discretize the hyperbolic saturation equation, we make use of an "Artificial Dissipation" (AD) method which was originally proposed by Jameson (1981), with the modifications introduced by Peraire *et al.* (1993). This method is based in the utilization of an adaptive artificial dissipative term that combines second order with fourth order diffusive terms. The basic idea of the method is to use the second order terms in regions of high gradients and to introduce the fourth order terms only in regions of smooth gradients in order to stabilize the scheme. Using the Green-Gauss theorem the semi-discrete numerical scheme for the solution of the saturation equation is written as

$$\phi \frac{\partial S_w}{\partial t} = -\frac{1}{V_I}\left(\sum_{L_{I(\Omega)}} \vec{F}^{\Omega}_{IJ_L(w)} \cdot \vec{C}_{IJ_L} + \sum_{L_{I(\Omega)}} \vec{F}^{\Gamma}_{IJ_L(w)} \cdot \vec{D}_{IJ_L} - Q_w V_I \right) \tag{9}$$

In Equation 9 the term $\vec{F}^{\Omega}_{IJ_L(w)} \cdot \vec{C}_{IJ_L}$ is replaced by

$$\vec{F}^{\Omega}_{IJ_L(w)} \cdot \vec{C}_{IJ_L} = \frac{1}{2}\left(\vec{F}_{I(w)} + \vec{F}_{J_L(w)} \right) \cdot \vec{C}_{IJ_L} + A.D. \tag{10}$$

This scheme has been proved reliable for a wide variety of fluid flow regimes. For further details of the computation of the AD term, see for instance Peraire *et al.* (1993), and Carvalho *et al.* (2005). Finally, time integration is performed through a simple two-level explicit time step scheme (Euler forward).

3.3 Handling Heterogeneities Using an Edge-Based Data Structure

Modeling fluid flow in highly heterogeneous porous media via a conservative formulation is a topic of intensive research (Edwards, 2002; Geiger *et al*., 2003). Using an edge-based data structure this subject becomes more interesting. Different strategies can be devised in order to properly approximate permeabilities in the middle of the edge, such as arithmetic average (resembling an edge-based finite element approach) or the utilization of an equivalent mid-edge permeability, in which the fluxes of a particular domain are obtained independently, i.e., the mid-edge velocities are computed using the correspondent domain permeability without using any explicit average. Further details can be found in Lyra *et al*. (2004). Even though this last approach had worked well for some heat transfer problems, (Lyra *et al*., 2004), it has been abandoned as it may lead to negative components for the equivalent mid-edge permeability tensor in the case of highly heterogeneous porous media. In the present paper we have adopted the traditional harmonic average. This choice is quite natural as it implies in the use of symmetric positive definite mid-edge tensors, besides, the use of harmonic averages with our compact stencil edge-based FV formulation, reduces the scheme to the classical two point conservative FD approach for 1-D problems.

4. Examples

4.1. 1/4 of Five Spot Problem with an Internal Low Permeability Zone

In this problem, adapted from Helmig (1997), there is a square low permeability zone between the injection and producer wells. The fluid and rock properties are $\rho_w = \rho_o = 1000\ kg/m^3$, $\mu_w = \mu_o = 0.001\ kg/(m.s)$, $\underset{\sim}{K} = 10^{-7}\underset{\sim}{I}\ m^2$ for zone 1, and $\underset{\sim}{K} = 10^{-10}\underset{\sim}{I}\ m^2$ for zone 2, and $\phi = 0.2$ in both regions. Therefore, permeability in zone 2 is smaller than in zone 1 by three orders of magnitude. Boundary conditions are $p = 2.10^5\ Pa$ and $S_w = 1$ at the injection well and $Q = -10.368\ m^3/d$ at the producer. Residual saturations are $S_{wr} = 0.0$ and $S_{or} = 0.0$. We also used a quadratic relative permeability-saturation relationship, i.e., $k(S_w) = \left((S_w - S_{wr})/(1 - S_{wr} - S_{or})\right)^2$. Two different unstructured triangular meshes were used for this example. The coarser one had 308 nodes, and the finer had 1180 nodes. The domain is a square of $L_1 = 300\ m$ and the low permeability zone is a square of $L_2 = 112.5\ m$. Figures 1a and 1b show respectively, the geometrical description of the problem and the saturation profile along the diagonal line connecting the wells (using the finer mesh). Figures 2a and 2b show, respectively, the saturation contours for the two different meshes at t = 520 days.

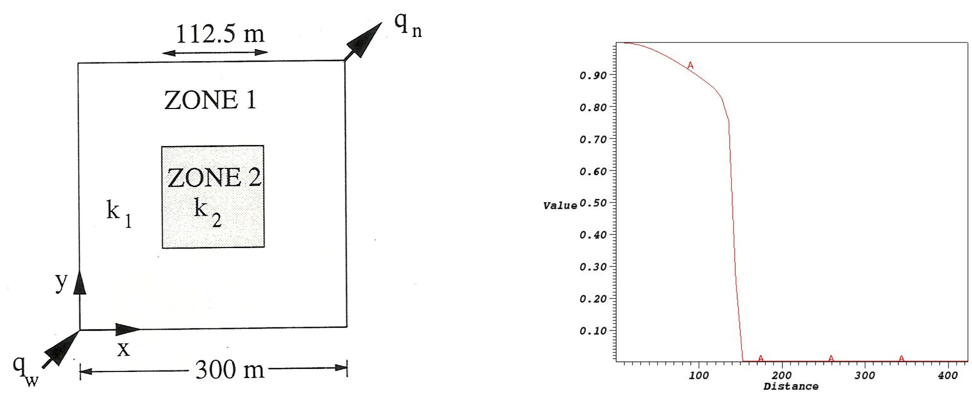
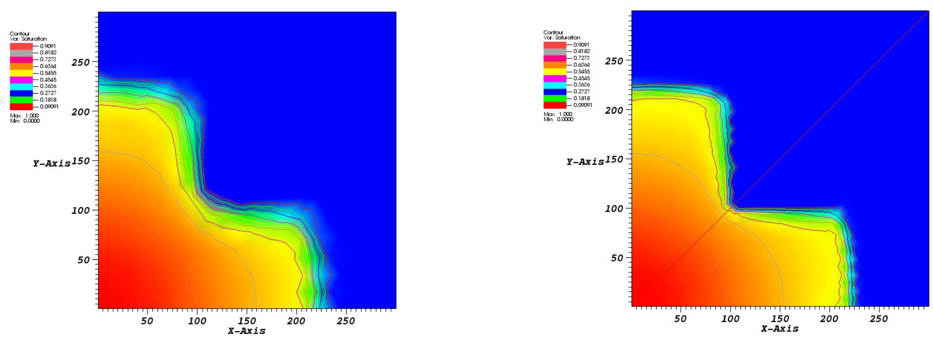


Figure 1. a) Geometrical description of the 1/4 of five spot problem with a low permeability zone; b) Saturation profile along line connecting the injector and producer wells at t =520 days.



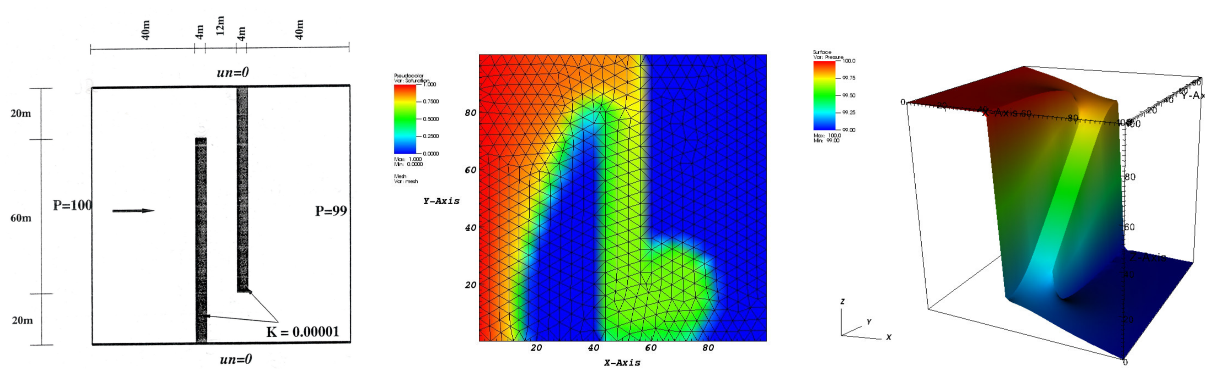
(a) (b)

Figure 2. Saturation contours for t =520 days using two different mesh densities: a) Coarser mesh with 308 nodes; b) Finer mesh with 1180 nodes.

The saturation profile of Figure 1b shows that, as expected, almost no fluid flow occurs through the low permeability zone. Indeed, as it can be seen in Figures 2a and 2b, even though the saturation contour for the coarser mesh is a little bit more diffusive, both meshes presented essentially the same behavior for the saturation field.

4.2. Confined Fluid Flow Between Perpendiculars Barriers

In this example, adapted from Garcia (1997), the fluid flow occurs due to the difference of pressure in a confined region that have two perpendicular barriers with extremely low permeability zones (permeability varies five orders of magnitude). These barriers form a channel in which fluid must preferentially flow. As in the first example, we have used a quadratic relative permeability relationship. Figure 3a shows the geometry and the problem data. Figures 3b and 3c show, respectively, the mesh and the saturation contour for t = 9500 days and the pressure surface contour (i.e., pressure field 'extruded' along Z-axis).



(a) (b) (c)

Figure 3. a) Geometry and physical data for problem 3.2; b) Saturation contour and unstructured mesh with 712 nodes and 1320 elements for t = 9500 days; c) Pressure surface showing the accurate modeling of the discontinuous gradient of pressure due to the existence of the perpendicular barriers.

It is clear from Figures 3b and 3c that the proposed FV formulation is capable of accurately solving, not only the pressure-velocity problem but also the saturation equation for this highly heterogeneous porous media, even with this relatively coarse mesh.

5. Conclusions

In the present paper we have briefly described an edge-based node-centered FV formulation solution of non-homogeneous two-dimensional two-phase flows in porous media. Some model examples were used to show the ability of the proposed formulation to properly solve fluid flow in highly heterogeneous porous media, such as typical oil reservoir formations. In the near future we intend to handle more realistic three-dimensional heterogeneous problems including capillarity and gravitational effects.

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References

BAOYAN, L, CHEN Z., GUAREN, H. Comparison of solution schemes for black oil reservoir simulations with unstructured grids. *Computer Methods in Appl. Mechs. Engrg.*, v. 193, p. 319-355, 2004.

CARVALHO, D. K. E., LYRA, P. R. M., WILLMERSDORF, R. B., ARAÚJO, F. D. S. An unstructured edge-based finite volume formulation for solving immiscible two-phase flows in porous media, *Communications in Numerical Methods in Engineering*, 2005 (in press).

CRUMPTON, P. I., MOINIER, P., GILES, M. B. T. J. An unstructured algorithm for high Reynolds number flows on highly stretched grids. In: C. Taylor and J. T. Cross, (editors), *Numerical Methods in Laminar and Turbulent Flow*, Pineridge Press, 561-572, 1997.

EDWARDS, M. G. Unstructured control-volume distributed, full-tensor finite volume schemes with flow based grids. *Computational Geociences*, v. 6, p. 433-452, 2002

GARCIA, E. L. M. Bi and tridimensional finite element formulations for the parallel simulation of fluid flows in petroleum reservoirs. Ph.D. Thesis (in portuguese), LNCC, Rio de Janeiro Brazil, 1997.

GEIGER, S., ROBERTS S., MATTHAI S. K., ZOPPOU, C. Combining finite volume and finite element methods to simulate fluid flow in geologic media. *ANZIAM J.* v. 44, pp. c180-c201, 2003.

HELMIG, R. *Multiphase Flow and Transport Processes in the Subsurface*. Springer, Berlin, 1997.

JAMESON, A., SHMIDT, W., TURKEL, E. Numerical simulation of the Euler equations by finite volume methods using runge-kutta time stepping schemes. *Technical Report, AIAA Paper*, p. 81-1259, 1981.

LUO, H., BAUM, J. D., LÖHNER, R. An improved finite volume scheme for compressible flows on unstructured grids. *Technical Report, AIAA Paper*, p. 95-0348, 1995.

LYRA, P. R. M., LIMA, R. C. F., GUIMARÃES, C. S. C., CARVALHO, D. K. E. An edge-based unstructured finite volume procedure for the numerical analysis of heat conduction applications. *Journal of the Brazilian Society of Mechanical Engineering, Brazil*, v. 26, p. 160-169, 2004.

PEACEMAN, D. W. *Fundamentals of Reservoir Simulation*. Elsevier, Amsterdam, 1977.

PERAIRE, J., PEIRÓ, J., MORGAN, K. Finite element multigrid solution of Euler flows past installed aero-engines. *J. Computational Mechanics*, v. 11, p. 433-451, 1993.

REES, I. MASTERS, I., MALAN, A. G., LEWIS, R. W. An edge-based finite volume scheme for saturated-unsaturated groundwater flow. *Comput. Methods Appl. Mech. Engrg.*, v. 193, p. 4741-4759, 2004.

SVÄRD, M., NORDSTRÖM, J. A Stable and Accurate Summation by Parts Finite Volume Formulation of the Laplacian Operator. *Uppsala Universitet, Institutionen för Informationsteknologi, Technical Report,* 003, 2003.